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Recent development on James constant of 2-dimensional Lorentz sequence spaces

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Abstract. The James constant of a Banach space X was introduced by Gao and Lau [3] and has recently been studied by several authors. In this paper, we present some recent results on James constant of 2-dimensional Lorentz sequence space.

1 Introduction

In the paper, we consider the James constant $J(X)$ of a Banach space X :

$$J(X) = \sup\{\min\{\|x + y\|, \|x - y\|\} : x, y \in S_X\},$$

where $S_X = \{x \in X : \|x\| = 1\}$. This notion was introduced by Gao and Lau [3] and recently it has been studied by several authors (cf. [5, 6, 17]). $\sqrt{2} \leq J(X) \leq 2$ for any Banach space X . In particular, $J(X) = \sqrt{2}$ if X is a Hilbert space. If $1 \leq p \leq \infty$ and $\dim L_p \geq 2$, then $J(L_p) = \max\{2^{1/p}, 2^{1/p'}\}$, where $1/p + 1/p' = 1$. $J(X) < 2$ holds if and only if X is uniformly non-square; that is, there exists a $\delta > 0$ such that $x, y \in S_X$ and $\|(x - y)/2\| \geq 1 - \delta$ imply $\|(x + y)/2\| \leq 1 - \delta$. Banach spaces with uniformly normal structure are characterized in terms of James constant (see [4, 5]). The Schäffer constant $g(X)$ of a Banach space X is defined by

$$g(X) = \inf\{\max\{\|x + y\|, \|x - y\|\} : x, y \in S_X\}.$$

We know that $1 \leq g(X) \leq \sqrt{2} \leq J(X) \leq 2$ and $g(X)J(X) = 2$ for all Banach spaces X .

A norm $\|\cdot\|$ on \mathbb{R}^2 is said to be absolute if $\|(z, w)\| = \|(|z|, |w|)\|$ for all $(z, w) \in \mathbb{R}^2$, and normalized if $\|(1, 0)\| = \|(0, 1)\| = 1$. The ℓ_p -norms $\|\cdot\|_p$ are such examples;

$$\|(z, w)\|_p = \begin{cases} (|z|^p + |w|^p)^{1/p} & \text{if } 1 \leq p < \infty, \\ \max\{|z|, |w|\} & \text{if } p = \infty. \end{cases}$$

Let AN_2 be the set of all absolute normalized norms on \mathbb{R}^2 , and Ψ_2 the set of all convex functions ψ on $[0, 1]$ satisfying $\max\{1-t, t\} \leq \psi(t) \leq 1$ ($0 \leq t \leq 1$). As in Bonsall and Duncan [1], AN_2 and Ψ_2 are in 1-1 correspondence under the equation

$$\psi(t) = \|(1-t, t)\| \quad (0 \leq t \leq 1). \quad (1)$$

Indeed, for all $\psi \in \Psi_2$ let

$$\|(z, w)\|_\psi = \begin{cases} (|z| + |w|)\psi\left(\frac{|w|}{|z| + |w|}\right) & \text{if } (z, w) \neq (0, 0), \\ 0 & \text{if } (z, w) = (0, 0). \end{cases}$$

Then $\|\cdot\|_\psi \in AN_2$, and $\|\cdot\|_\psi$ satisfies (1). From this result, we can consider many non ℓ_p -type norms easily. Now let $\psi_p(t) = \{(1-t)^p + t^p\}^{1/p} \in \Psi_2$. As is easily seen, the ℓ_p -norm $\|\cdot\|_p$ is associated with ψ_p .

In this note we present some recent results about James constants of two dimensional Lorentz sequence space $d^{(2)}(\omega, q)$ and its dual $d^{(2)}(\omega, q)^*$. In general, $J(X) \neq J(X^*)$ for a space X and its dual X^* , whereas we will obtain $J(d^{(2)}(\omega, q)) = J(d^{(2)}(\omega, q)^*)$ for all ω, q .

2 James constant

We first discuss James constant of absolute norms. Let X be a Banach space and $x, y \in X$. We say that x is isosceles orthogonal to y , denoted by $x \perp_I y$, if $\|x+y\| = \|x-y\|$. We define a function $\beta(x)$ on X by

$$\beta(x) = \sup\{\min\{\|x+y\|, \|x-y\|\} : y \in S_X\}.$$

To obtain $J((\mathbb{R}^2, \|\cdot\|_\psi))$, we need the following lemma.

Lemma 1 ([3]). *Let $\psi \in \Psi_2$ and $x \in S_{(\mathbb{R}^2, \|\cdot\|_\psi)}$. Then there exists uniquely a vector $y_0 \in S_{(\mathbb{R}^2, \|\cdot\|_\psi)}$ with $x \perp_I y_0$. Moreover, $\beta(x) = \|x + y_0\|_\psi$.*

From this lemma, we can write

$$J((\mathbb{R}^2, \|\cdot\|_\psi)) = \sup\{\|x+y\|_\psi : x, y \in S_{(\mathbb{R}^2, \|\cdot\|_\psi)}, x \perp_I y\}.$$

We recall that an absolute normalized norm $\|\cdot\|$ on \mathbb{R}^2 is symmetric in the sense that $\|(x, y)\| = \|(y, x)\|$ for all $(x, y) \in \mathbb{R}^2$ if and only if the corresponding function ψ is symmetric with respect to $t = 1/2$, that is, $\psi(1-t) = \psi(t)$ for every $t \in [0, 1]$. Using Lemma 1 we can obtain the following formula by using ψ in Ψ_2 .

Theorem 2 ([11]). *Let $\psi \in \Psi_2$. If ψ is symmetric with respect to $t = 1/2$, then*

$$J((\mathbb{R}^2, \|\cdot\|_\psi)) = \max_{0 \leq t \leq 1/2} \frac{2-2t}{\psi(t)} \psi\left(\frac{1}{2-2t}\right).$$

Corollary 3 ([11]). *Let $\psi \in \Psi_2$. Assume that ψ is symmetric with respect to $t = 1/2$.*

(i) *If $\psi \geq \psi_2$ and $M_1 = \max_{0 \leq t \leq 1} \psi(t)/\psi_2(t)$ is taken at $t = 1/2$, then*

$$J((\mathbb{R}^2, \|\cdot\|_\psi)) = 2\psi\left(\frac{1}{2}\right).$$

(ii) *If $\psi \leq \psi_2$ and $M_2 = \max_{0 \leq t \leq 1} \psi_2(t)/\psi(t)$ is taken at $t = 1/2$, then*

$$J((\mathbb{R}^2, \|\cdot\|_\psi)) = \frac{1}{\psi(1/2)}.$$

Example 1. Let $1 \leq p \leq \infty$ and $1/p + 1/p' = 1$.

(i) If $1 \leq p \leq 2$, then $J((\mathbb{R}^2, \|\cdot\|_p)) = 2\psi_p(1/2) = 2^{1/p}$.

(ii) If $2 \leq p \leq \infty$, then $J((\mathbb{R}^2, \|\cdot\|_p)) = 1/\psi_p(1/2) = 2^{1/p'}$.

For $0 < \omega < 1$ and $1 \leq q < \infty$, the 2-dimensional Lorentz sequence space $d^{(2)}(\omega, q)$ is \mathbb{R}^2 with the norm

$$\|x\|_{\omega, q} = (x_1^{*q} + \omega x_2^{*q})^{1/q}, \quad x = (x_1, x_2) \in \mathbb{R}^2,$$

where (x_1^*, x_2^*) is the nonincreasing rearrangement of $(|x_1|, |x_2|)$; that is, $x_1^* = \max\{|x_1|, |x_2|\}$ and $x_2^* = \min\{|x_1|, |x_2|\}$. Note here that the norm $\|\cdot\|_{\omega, q}$ of $d^{(2)}(\omega, q)$ is a symmetric absolute normalized norm on \mathbb{R}^2 , and the corresponding convex function is given by

$$\psi_{\omega, q}(t) = \begin{cases} ((1-t)^q + \omega t^q)^{1/q} & \text{if } 0 \leq t \leq 1/2, \\ (t^q + \omega(1-t)^q)^{1/q} & \text{if } 1/2 \leq t \leq 1. \end{cases}$$

Kato and Maligranda in [6] computed the James constant of $d^{(2)}(\omega, q)$ in the case where $q \geq 2$.

Theorem 4 ([6]). *Let $q \geq 2$ and $0 < \omega < 1$. Then*

$$J(d^{(2)}(\omega, q)) = 2 \left(\frac{1}{1+\omega} \right)^{1/q}.$$

For the case where $q < 2$, we attempt to calculate it and we had a partial answer in Mitani and Saito [11] and Suzuki, Yamano and Kato [16] by using Theorem 2. In Mitani, Saito and Suzuki [12] we have a complete answer.

Theorem 5 ([11]). *Let $q = 1$. If $0 < \omega \leq \sqrt{2} - 1$, then*

$$J(d^{(2)}(\omega, q)) = \frac{2}{1 + \omega}.$$

If $\sqrt{2} - 1 < \omega < 1$, then

$$J(d^{(2)}(\omega, q)) = 1 + \omega.$$

Theorem 6 ([12, 14]). *Let $1 < q < 2$. (i) If $0 < \omega \leq (\sqrt{2} - 1)^{2-q}$, then*

$$J(d^{(2)}(\omega, q)) = 2 \left(\frac{1}{1 + \omega} \right)^{1/q}$$

and

$$g(d^{(2)}(\omega, q)) = (1 + \omega)^{1/q}.$$

(ii) *If $(\sqrt{2} - 1)^{2-q} < \omega < 1$, then there exists a unique pair of real numbers s_0, s_1 such that*

$$\left(\frac{1 - \omega}{\omega(1 + \omega)} \right)^{p-1} < s_0 < \omega^{1/(2-q)} < s_1 < 1$$

and $(1 + s_i)^{q-1}(1 - \omega s_i^{q-1}) = \omega(1 - s_i)^{q-1}(1 + \omega s_i^{q-1})$ for $i = 0, 1$.

(ii-a) *If $(\sqrt{2} - 1)^{2-q} < \omega \leq \sqrt{2}^q - 1$, then*

$$J(d^{(2)}(\omega, q)) = \max \left\{ \left(\frac{2(1 + s_0)^{q-1}}{1 + \omega s_0^{q-1}} \right)^{1/q}, 2 \left(\frac{1}{1 + \omega} \right)^{1/q} \right\}$$

and

$$g(d^{(2)}(\omega, q)) = \min \left\{ \left(\frac{2(1 + s_1)^{q-1}}{1 + \omega s_1^{q-1}} \right)^{1/q}, (1 + \omega)^{1/q} \right\}.$$

(ii-b) *If $\sqrt{2}^q - 1 < \omega < 1$, then*

$$J(d^{(2)}(\omega, q)) = \left(\frac{2(1 + s_0)^{q-1}}{1 + \omega s_0^{q-1}} \right)^{1/q}$$

and

$$g(d^{(2)}(\omega, q)) = \left(\frac{2(1 + s_1)^{q-1}}{1 + \omega s_1^{q-1}} \right)^{1/q}.$$

We next consider the dual space of $d^{(2)}(\omega, q)$. For $q = 1$, it is known that $d^{(2)}(\omega, 1)^*$ is a 2-dimensional Marcinkiewicz space m_ω given by the norm

$$\|(x, y)\|_{m_\omega} = \max \left\{ x^*, \frac{x^* + y^*}{1 + \omega} \right\},$$

where $x^* = \max\{|x|, |y|\}$, $y^* = \min\{|x|, |y|\}$. For $\psi \in \Psi_2$ let $\|\cdot\|_\psi^*$ be the dual of the norm $\|\cdot\|_\psi$. Namely, $\|x\|_\psi^* = \sup\{|\langle x, y \rangle| : y \in S_{(\mathbb{R}^2, \|\cdot\|_\psi)}\}$ for any $x \in \mathbb{R}^2$. Then $\|\cdot\|_\psi^* \in AN_2$ and the corresponding convex function ψ^* in Ψ_2 is

$$\psi^*(t) = \sup_{0 \leq s \leq 1} \frac{(1-s)(1-t) + st}{\psi(s)}$$

for t with $0 \leq t \leq 1$.

Theorem 7 ([13]). *Let $0 < \omega < 1$. (i) If $1 < q < \infty$, then*

$$\psi_{\omega, q}^*(t) = \begin{cases} ((1-t)^p + \omega^{1-p}t^p)^{1/p}, & \text{if } 0 \leq t < \frac{\omega}{1+\omega}, \\ (1+\omega)^{1/p-1}, & \text{if } \frac{\omega}{1+\omega} \leq t < \frac{1}{1+\omega}, \\ (t^p + \omega^{1-p}(1-t)^p)^{1/p}, & \text{if } \frac{1}{1+\omega} \leq t \leq 1, \end{cases}$$

where $1/p + 1/q = 1$.

(ii) *If $q = 1$, then*

$$\psi_{\omega, 1}^*(t) = \begin{cases} 1-t, & \text{if } 0 \leq t < \frac{\omega}{1+\omega}, \\ \frac{1}{1+\omega}, & \text{if } \frac{\omega}{1+\omega} \leq t < \frac{1}{1+\omega}, \\ t, & \text{if } \frac{1}{1+\omega} \leq t \leq 1. \end{cases}$$

Hence,

Theorem 8 ([13]). *Let $0 < \omega < 1$. (i) If $1 < q < \infty$, then*

$$\|(x, y)\|_{\omega, q}^* = \begin{cases} (|x|^p + \omega^{1-p}|y|^p)^{1/p} & \text{if } \omega|x| \geq |y|, \\ (1+\omega)^{1/p-1}(|x| + |y|) & \text{if } \omega|x| \leq |y| \leq \omega^{-1}|x|, \\ (|y|^p + \omega^{1-p}|x|^p)^{1/p} & \text{if } \omega^{-1}|x| \leq |y|. \end{cases}$$

(ii) *If $q = 1$, then*

$$\|(x, y)\|_{\omega, 1}^* = \begin{cases} \max\{|x|, \omega^{-1}|y|\} & \text{if } \omega|x| \geq |y|, \\ \frac{1}{1+\omega}(|x| + |y|) & \text{if } \omega|x| \leq |y| \leq \omega^{-1}|x|, \\ \max\{\omega^{-1}|x|, |y|\} & \text{if } \omega^{-1}|x| \leq |y|. \end{cases}$$

Namely, $\|(x, y)\|_{\omega, 1}^* = \|(x, y)\|_{m_\omega}$.

We consider the constant $J(d^{(2)}(\omega, q)^*)$ for ω, q .

Theorem 9 ([13]). (i) Let either $q \geq 2$ and $0 < \omega < 1$, or $1 < q < 2$ and $0 < \omega \leq (\sqrt{2} - 1)^{2-q}$. Then

$$J(d^{(2)}(\omega, q)^*) = 2 \left(\frac{1}{1+\omega} \right)^{1/q}.$$

(ii) Let $q = 1$. If $0 < \omega \leq \sqrt{2} - 1$, then

$$J(d^{(2)}(\omega, q)^*) = \frac{2}{1+\omega}.$$

If $\sqrt{2} - 1 < \omega < 1$, then

$$J(d^{(2)}(\omega, q)^*) = 1 + \omega.$$

Hence $J(d^{(2)}(\omega, q))$ and $J(d^{(2)}(\omega, q)^*)$ coincide for such cases. In the case where $1 < q < 2$ and $\omega > (\sqrt{2} - 1)^{2-q}$, we recently obtained the following result.

Theorem 10 ([13, 14]). Let $1 < q < 2$ and $1/p + 1/q = 1$. If $(\sqrt{2} - 1)^{2-q} < \omega < 1$, then there exists a unique pair of real numbers s_0, s_1 such that

$$\left(\frac{1 - \omega}{\omega(1 + \omega)} \right)^{p-1} < s_0 < \omega^{1/(2-q)} < s_1 < 1$$

and $(1 + s_i)^{q-1}(1 - \omega s_i^{q-1}) = \omega(1 - s_i)^{q-1}(1 + \omega s_i^{q-1})$ for $i = 0, 1$.

(i) If $(\sqrt{2} - 1)^{2-q} < \omega \leq \sqrt{2}^q - 1$, then

$$J(d^{(2)}(\omega, q)^*) = \max \left\{ \left(\frac{2(1 + s_0)^{q-1}}{1 + \omega s_0^{q-1}} \right)^{1/q}, 2 \left(\frac{1}{1 + \omega} \right)^{1/q} \right\}$$

and

$$g(d^{(2)}(\omega, q)^*) = \min \left\{ \left(\frac{2(1 + s_1)^{q-1}}{1 + \omega s_1^{q-1}} \right)^{1/q}, (1 + \omega)^{1/q} \right\}.$$

(ii) If $\sqrt{2}^q - 1 < \omega < 1$, then

$$J(d^{(2)}(\omega, q)^*) = \left(\frac{2(1 + s_0)^{q-1}}{1 + \omega s_0^{q-1}} \right)^{1/q}$$

and

$$g(d^{(2)}(\omega, q)^*) = \left(\frac{2(1 + s_1)^{q-1}}{1 + \omega s_1^{q-1}} \right)^{1/q}.$$

Thus,

Theorem 11 ([14]). *If $1 \leq q < \infty$ and $0 < \omega < 1$, then*

$$J(d^{(2)}(\omega, q)^*) = J(d^{(2)}(\omega, q))$$

and

$$g(d^{(2)}(\omega, q)^*) = g(d^{(2)}(\omega, q)).$$

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